PMM U.S.S.R.,Vol.53,No.4,pp.411-416,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

THE CONSTRUCTION OF PHASE PATHS OF A HAMILTONIAN SYSTEM IN THE NEIGHBOURHOOD OF AN EQUILIBRIUM*

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A non-autonomous periodic Hamiltonian system with one degree of freedom is studied in the neighbourhood of an elliptical equilibrium point. A uniform approximation of the solution in a finite time interval in the resonance case is determined using the projection method, instead of the traditional perturbation theoretical method.

The Cauchy problem is reduced to a functional equation in the space of the derivatives, and a Galerkin scheme is constructed for this equation. A theorem is proved on convergence of the sequence of approximations to the exact solution. Every finite-dimensional approximation of sufficiently high order may be found by explicit iterations. The results can be generalized to dynamical systems of higher dimensions.

1. Statement of the problem. Near the equilibrium $\mathbf{q}=\mathbf{p}=0$, the canonical system with n degrees of freedom has the form

$$\mathbf{q}^{\cdot}=H_{\mathbf{p}},\ \mathbf{p}^{\cdot}=-H_{\mathbf{q}}\quad(\mathbf{q},\mathbf{p}\in\mathbf{R}^n) \tag{1.1}$$

where the Hamilton function can be expanded in a power series around zero starting with second-order terms

$$H(\mathbf{q},\mathbf{p},t)=H_2(\mathbf{q},\mathbf{p},t)+H_3(\mathbf{q},\mathbf{p},t)+\ldots$$

The paths in a small neighbourhood can be constructed by perturbation theory. Changing to new variables $\mathbf{q}=\varepsilon\mathbf{x}$, $\mathbf{p}=\varepsilon\mathbf{y}$, we obtain the system of equations

$$\mathbf{x}^{\cdot}=K_{\mathbf{y}},\ \mathbf{y}^{\cdot}=-K_{\mathbf{x}}\ (\mathbf{x},\mathbf{y}\in\mathbf{R}^n)$$

The new Hamiltonian has the form

$$K(\mathbf{x},\mathbf{y},t,\varepsilon)=H_2(\mathbf{x},\mathbf{y},t)+\varepsilon H_3(\mathbf{x},\mathbf{y},t)+\ldots \tag{1.2}$$

The non-perturbed case corresponds to a linear (in general non-autonomous) system. We can change to new variables by the method of variation of arbitrary constants. Denoting the phase vector by $\mathbf{z}=(\mathbf{x},\mathbf{y})^T$, we can write the solution of the Hamiltonian system in the form

$$\mathbf{z}=Z(t)\,\zeta\quad(Z(0)=E,\ \zeta\in\mathbf{R}^{2n}) \tag{1.3}$$

where $Z(t)$ is the fundamental matrix. The transformation defined by $Z(t)$ is obviously canonical. Changing to the phase vector ζ, we obtain the Hamiltonian system

$$\begin{gathered}\zeta^{\cdot}=\varepsilon IF_{\zeta}(\zeta,t,\varepsilon)\quad(I^2=-E)\\ \varepsilon F(\zeta,t,\varepsilon)=\varepsilon H_3(Z(t)\,\zeta,t)+\ldots\end{gathered} \tag{1.4}$$

The procedure proposed in this paper can be applied after the given problem has been reduced to this form.

To fix our ideas, consider a system with one degree of freedom and a 2π-periodic (in time) Hamilton function $K(q,p,t)$ $(q,p\in\mathbf{R})$. The problem of the motion of an asteroid in the neighbourhood of a periodic orbit can be reduced to such a problem. Thus, let $\pm i\lambda\neq0$ be the characteristic exponents of the first-approximation system, where 2λ is a non-integer. Then the canonical 2π-periodic transformation $(q,p)\mapsto(Q,P)$ reduces H_2 to normal form $H_2=2^{-1}\lambda(Q^2+P^2)$.

We apply a scaling transformation to enlarge the neighbourhood of the equilibrium as described above: $Q=\varepsilon x$, $P=\varepsilon y$. The solution of the unperturbed problem now has the form (1.3), where $\mathbf{z}=(x,y)^T$, and $Z(t)$ is the fundamental matrix

*Prikl.Matem.Mekhan.,53,4,531-538,1989

$$Z(t)=\left\|\begin{matrix}\cos\lambda t & \sin\lambda t\\ -\sin\lambda t & \cos\lambda t\end{matrix}\right\|$$

Thus,

$$\mathbf{z}(t)=\zeta\cos\lambda t+I\zeta\sin\lambda t=2^{-1}(\zeta-iI\zeta)e^{i\lambda t}+(2i)^{-1}(\zeta+iI\zeta)e^{-i\lambda t}$$

where $\zeta=(\zeta_1,\zeta_2)$, and I is the symplectic matrix $(I^2=-E)$. Thus, the system of differential equations has a standard form (1.4) (in the sense of Bogolyubov /1/) with the Hamilton function

$$\varepsilon F(\zeta,t,\varepsilon)=\varepsilon H_3(2^{-1}(\zeta-iI\zeta)e^{i\lambda t}+(2i)^{-1}(\zeta+iI\zeta)e^{-i\lambda t},t)+\ldots$$

Since the explicit dependence of the homogeneous forms $H_k(\mathbf{z},t)$ $(k\geqslant 3)$ on t is 2π-periodic, their coefficients can be expanded in Fourier series in the functions e^{int} $(n\in\mathbf{Z})$. These expansions may be treated as Laurent series in the variable e^{it}. The Hamiltonian can be conveniently represented in the form

$$\varepsilon F(\zeta,t,\varepsilon)=\sum_{s=3}^{\infty}\varepsilon^{s-2}F_s(\zeta,e^{i\lambda t},e^{it})$$

where the homogeneous forms F_s (in the variables ζ_i $(i=1,2)$ are expressed as $F_s=H_s(Z(t)\zeta,t)$.

Resonance cases are of particular interest. In these cases, we should have $k\lambda=r\in\mathbf{Z}$ $(k\geqslant 3)$ and the Hamilton function $\varepsilon F(\zeta,t,\varepsilon)$ is t-periodic. For ease of presentation, we will change to a new independent variable τ by the formula $t=k\tau$. The homogeneous forms F_s are polynomial in ζ, $e^{ir\tau}$, and $e^{ik\tau}$, and their expansions contain both positive and negative powers of exponential functions.

In what follows, the phase space is $\mathbf{C}^2$ - the complexification of $\mathbf{R}^2$.

After all these transformations, we obtain a non-autonomous τ-periodic system of second-order differential equations (the prime denotes differentiation with respect to τ)

$$\zeta'=\varepsilon\mathbf{Z}(\zeta,\tau,\varepsilon)\tag{1.5}$$

$$\varepsilon\mathbf{Z}(\zeta,\tau,\varepsilon)=\varepsilon kIF_\zeta(\zeta,k\tau,\varepsilon)=\sum_{s=3}^{\infty}\varepsilon^{s-2}kIF_{s\zeta}(\zeta,e^{ir\tau},e^{ik\tau})=\sum_{s=3}^{\infty}\varepsilon^{s-2}\mathbf{Z}_s(\zeta,\tau)\tag{1.6}$$

The vector functions $\mathbf{Z}_s$ are homogeneous in ζ of degree s.

2. ***Reduction.*** Our goal now is construct in $[0,2\pi]$ the solution of the Cauchy problem of system (1.5) corresponding to the initial condition vector ζ_0. We will use the projection method. In order to ensure uniform approximation to the solution in $[0,2\pi]$, further transformation of the problem is required. We will change from the space of continuous vector functions $\zeta(\tau)$ to the space of the derivatives $\zeta'(\tau)$.

Let us formalize our statements in rigorous form. We denote by CA the class of functions ζ: $[a,b]\to\mathbf{C}^n$ absolutely continuous in $[a,b]\subset\mathbf{R}$. CA is a linear space, and if we introduce the norm

$$\|\zeta\|_A=\|\zeta(a)\|+\mathrm{Var}([a,b],\zeta)\tag{2.1}$$

then CA becomes a Banach space.

On the other hand, consider the Banach space L_1 of classes of almost everywhere equal Lebesgue-integrable functions γ: $[a,b]\to\mathbf{C}^n$. The norm in L_1 is defined by the formula

$$\|\gamma\|_1=\int_a^b\|\gamma(\tau)\|\,d\tau\tag{2.2}$$

Now let D: $CA\to L_1$ be the differentiation operator with respect to the variable τ: $(D\zeta)(\tau)=\zeta'(\tau)$. We know from analysis /2/ that there is a unique correspondence between the properties of summability and absolute continuity of functions of a real variable. If the function $\zeta(\tau)$ is absolutely continuous, then $\zeta'(\tau)$ is summable in $[a,b]$ and, conversely, if $\gamma\in L_1$, then the function

$$\zeta(\tau)=(D^{-1}\gamma)(\tau)=\zeta_0+\int_a^\tau\gamma(\alpha)\,d\alpha\tag{2.3}$$

is absolutely continuous. The operator D^{-1} is uniquely defined if the vector $\zeta_0 \in \mathbf{C}^n$ is fixed.

In what follows, we consider solutions of systems of differential equations of the form (1.5). Let Q be a neighbourhood of an equilibrium where the vector function $\mathbf{Z}(\zeta, \tau, \varepsilon)$ is defined for all $\tau \in [a, b]$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Theorem 1. If $\zeta_0 \in Q$ is fixed, then $\zeta \in CA$ is the solution of the Cauchy problem for Eqs.(1.5) in $[a, b]$ if and only if $\gamma = D\zeta$ is the solution of the functional equation

$$\gamma = \Phi(\gamma) \tag{2.4}$$

in the space L_1, with the non-linear operator Φ defined by the formula

$$[\Phi(\gamma)](\tau) = \varepsilon \mathbf{Z}[(D^{-1}\gamma)(\tau), \tau, \varepsilon] \tag{2.5}$$

We will apply the projection method to Eq.(2.4). To this end, we have to pass from the space L_1 to a more restricted (Hilbert) space L_2 of classes of functions $\gamma: [a, b] \to \mathbf{C}^n$ with the Hermitian product

$$\langle \gamma_1, \gamma_2 \rangle_2 = \int_a^b \langle \gamma_1(\tau), \gamma_2(\tau) \rangle \, d\tau \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian product in $\mathbf{C}^n$. The norm in L_2 is defined by the formula

$$\|\gamma\|_2 = \left(\int_a^b \|\gamma(\tau)\|^2 \, d\tau \right)^{1/2} \tag{2.7}$$

We know that when the interval $[a, b]$ is finite, the space L_2 is continuously embedded in L_1. Therefore, its pre-image $D^{-1}(L_2)$ for a fixed ζ_0 is also continuously embedded in CA, which follows from the continuity of the operators D and D^{-1}. In symmetry, we have the following proposition.

Theorem 2. If the sequences $\{\gamma_s\}_{s=1}^{\infty}$ converge to the solution γ of Eq.(2.4) in the space L_2, then the sequence $\{D^{-1}\gamma_s\}_{s=1}^{\infty}$ converges to the solution $D^{-1}\gamma$ of Eq.(1.5) in the space CA, and in particular the convergence is uniform.

3. ***Approximation theorem***. Let $\{\mathbf{e}_j\}_{j=1}^{n}$ be an orthonormal basis in $\mathbf{C}^n$ and $\{g_m(\tau)\}_{m=1}^{\infty}$ an orthonormal basis in $L_2([a, b], \mathbf{C})$. Then all the vector functions $g_m(\tau)\mathbf{e}_j = \psi_{jm}(\tau)$ ($j = 1, 2, \ldots, n$; $m = 0, \pm 1, \pm 2, \ldots$) form an orthonormal basis in L_2. Linearly ordering the system of functions $\{\psi_{jm}\}$, we obtain the orthonormal basis $\{\varphi_m\}_{m=1}^{\infty}$. For the case of the interval $[0, 2\pi]$, the basis $\{g_m(\tau)\}$ is conveniently chosen as the trigonometric system of functions $\{(2\pi)^{-1/2} e^{im\tau}\}$ $(m \in \mathbf{Z})$.

Let P_m $(m \in \mathbf{N})$ be the orthogonal projection operator on the finite-dimensional space spanned by the first m basis vectors $\{\varphi_s\}_{s=1}^{m}$.

The right-hand sides of Eq.(1.5) are usually assumed to be sufficiently smooth. In our case, we have analyticity in the variables ζ. Therefore, the condition $\mathbf{Z} \in L_2([a, b], C^1(Q))$ or

$$\|\mathbf{Z}\|^1_2 = \left\{ \int_a^b [\sup_{\zeta \in Q} (\|\mathbf{Z}(\zeta, \tau, \varepsilon)\|, \|\mathbf{Z}'_\zeta(\zeta, \tau, \varepsilon)\|)]^2 \, d\tau \right\}^{1/2} < \infty \tag{3.1}$$

used below is sufficiently weak and a priori satisfied.

By the existence and uniqueness of the solution of the Cauchy problem $\zeta(\tau)$ for Eq. (1.5), Eq.(2.4) should also have a unique solution $\zeta'(\tau)$ (by Theorem 1). It remains to construct the Galerkin approximations $\gamma_m(\tau)$ of this function in the space L_2. They are solutions of the (in general non-linear) finite-dimensional equations

$$\gamma_m = P_m \Phi(\gamma_m) \quad (\gamma_m \in P_m L_2, m \in \mathbf{N}) \tag{3.2}$$

We will apply the result of /3/ to the functional Eq.(2.4). This requires refining the domain of definition of the operator Φ. This is the set $\Omega \subset L_2$ of functions $\gamma(\tau)$ such that for a fixed $\zeta_0 \in Q$ for all $\tau \in [a, b]$ we have $(D^{-1}\gamma)(\tau) \in Q$. Since $Q \subset \mathbf{C}^n$ is open and the value set of the vector function $(D^{-1}\gamma)(\tau)$ is compact (because the interval $[a, b]$ is finite), we obtain that the set Ω is open in the space L_2. Thus, Ω is a domain in L_2.

Theorem 3. When condition (3.1) holds for a fixed $\zeta_0 \in Q$, Eq.(2.4) has a unique solution γ° if the solution of the Cauchy problem of Eq.(1.5) exists in the entire interval.

Moreover, if $\varepsilon > 0$ is sufficiently small, then there exist an integer N and $\delta > 0$ such that for any $m > N$ Eq.(3.2) has a unique solution γ_m in the sphere $\| \gamma - \gamma^\circ \|_2 \leqslant \delta$ and

$$\| \gamma_m - \gamma^\circ \|_2 \leqslant \| \gamma^\circ - P_m \gamma^\circ \|_2 + \| \gamma_m - P_m \gamma^\circ \|_2 \to 0 \quad (m \to \infty)$$

and for some $c_1, c_2 > 0$ we also have the two-sided bound

$$c_1 \| P_m \Phi(\gamma^\circ) - P_m \Phi(P_m \gamma^\circ) \|_2 \leqslant \| \gamma_m - P_m \gamma^\circ \|_2 \leqslant c_2 \| P_m \Phi(\gamma^\circ) - P_m \Phi(P_m \gamma^\circ) \|_2$$

This theorem guarantees that the finite-dimensional solution $\gamma_m(\tau)$ obtained by the Galerkin scheme (3.2) is an approximation in L_2 to the exact solution $\gamma^\circ(\tau)$ of Eq.(2.4).

Suppose that the function γ_m has been found in the form

$$\gamma_m(\tau) = \sum_{j=1}^{m} \gamma_m{}^j \varphi_j(\tau) \quad (\gamma_m{}^j \in \mathbf{C}) \tag{3.3}$$

Then by Theorem 2 the vector function

$$\zeta_m(\tau) = \zeta_0 + \sum_{j=1}^{m} \gamma_m{}^j \int_a^\tau \varphi_j(\alpha)\, d\alpha$$

is a uniform approximation to the required solution of the Cauchy problem.

4. Approximation algorithm. Returning to the two-dimensional system of differential Eqs. (1.5), consider an orthonormal basis in the vector functions space $L_2([0, 2\pi], \mathbf{C}^2)$. It is defined by a trigonometric orthonormal system and consists of vector functions of the form $(2\pi)^{-1/2} e^{is\tau} \mathbf{e}_j$ $(j = 1, 2;\ s \in \mathbf{Z})$. We number the basis functions so that

$$\varphi_m(\tau) = (2\pi)^{-1/2} e^{is\tau} \mathbf{e}_j \ (j = m - 2\,[(m-1)/2], \tag{4.1}$$
$$s = [s'/2]\,(2\,(s' - 2\,[s'/2]) - 1),\ s' = [(m-1)/2] + 1,\ m \in \mathbf{N})$$

In the Galerkin scheme we use the projectors P_{4m+2}. Then the finite-dimensional approximations to the exact solution lie in the space formed by the functions $(2\pi)^{-1/2} e^{is\tau} \mathbf{e}_j$ $(j = 1, 2;\ s = 0, \pm 1, \pm 2, \ldots)$.

We recall that the basis vectors in $\mathbf{C}^2$ may be treated as coordinate columns $\mathbf{e}_1 = (1, 0)^T$, $\mathbf{e}_2 = (0, 1)^T$. Then any function in the space $P_{4m+2} L_2([0, 2\pi], \mathbf{C}^2) \simeq \mathbf{C}^{4m+2}$ can be represented in the form

$$\gamma_m(\tau) = \Sigma_m e^{is\tau} \mathbf{c}_s \quad (\mathbf{c}_s = (c_s{}^1, c_s{}^2)^T) \tag{4.2}$$

where $\mathbf{c}_s \in \mathbf{C}^2$ are arbitrary complex column vectors. Here and henceforth, Σ_m stands for summation over s from $s = -m$ to $s = m$.

Consider the right-hand side of system (1.5) defined by relationship (1.6). The homogeneous forms $\mathbf{Z}_s(\zeta, \tau)$ with periodic coefficients can be represented in terms of symmetrical s-linear forms $L_s(\zeta_1, \zeta_2, \ldots, \zeta_s, \tau)$. Therefore $\mathbf{Z}_s(\zeta, \tau) = L_s(\zeta, \zeta, \ldots, \zeta, \tau)$ (see equality (1.6)).

In order to derive an equation of the form (3.2), we apply the operator D^{-1} to the function (4.2), which gives

$$(D^{-1}\gamma_m)(\tau) = \zeta_0 - \Sigma_m{}' \frac{1}{is} \mathbf{c}_s + \Sigma_m{}' \frac{1}{is} e^{is\tau} \mathbf{c}_s + \tau \mathbf{c}_0 \tag{4.3}$$

Here and henceforth, $\Sigma_m{}'$ denotes summation over s from $s = -m$ to $s = -1$ and from $s = +1$ to $s = m$.

After substituting this formula and the Fourier expansions of the periodic coefficients of the forms L_s into the right-hand side of (1.6), we obtain a function of the form

$$\varepsilon \mathbf{Z}(\zeta, \tau, \varepsilon) = \varepsilon \Sigma_\infty (\mathbf{f}_{0s}(\mathbf{c}, \varepsilon) + \mathbf{f}_{1s}(\mathbf{c}, \varepsilon)\tau + \mathbf{f}_{2s}(\mathbf{c}, \varepsilon)\tau^2) e^{is\tau} + \sum_{j=3}^{\infty} \varepsilon^{j-1} \tau^j \Sigma_\infty \mathbf{f}_{js}(\mathbf{c}, \varepsilon) e^{is\tau} \tag{4.4}$$

$\mathbf{c} = (c_0{}^1, c_0{}^2, c_{-1}{}^1, c_{-1}{}^2, c_1{}^1, c_1{}^2, \ldots, c_m{}^1, c_m{}^2)^T$ ($\mathbf{c}$ is the vector of unknown coefficients). The functions $\mathbf{f}_{js}(\mathbf{c}, \varepsilon)$ $(j = 0, 1, \ldots;\ s = -\infty, \ldots, +\infty)$ are analytic in $\mathbf{c}$ and ε. Power series expansions can be obtained after collecting similar terms with products of the form $\tau^j e^{is\tau}$.

Representing the functions $\tau^j e^{is\tau}$ as Fourier series in $[0, 2\pi]$ and substituting these expansions into (4.4), we obtain the right-hand side in the form

$$\varepsilon \mathbf{Z}(\zeta, \tau, \varepsilon) = \varepsilon \Sigma_\infty \mathbf{h}_s(\mathbf{c}, \varepsilon) e^{is\tau} \tag{4.5}$$

Note that the products $\tau e^{is\tau}$ occur on the right-hand side with a multiplier ε^{j-1}. This enables us to take into account perturbations in the procedure of the expansion of the right-hand side, because for sufficiently small $\varepsilon > 0$ the product $\varepsilon^{j-1}\tau^j$ tends to zero as $j \to \infty$.

Applying the projection operators P_{4m+2} to the function (4.5), we obtain the Galerkin equation ($\mathbf{h}(\mathbf{c}, \varepsilon)$ is an analytical vector function)

$$\mathbf{c} = \varepsilon \mathbf{h}(\mathbf{c}, \varepsilon) \tag{4.6}$$

Eq.(4.6) can be solved iteratively by the formula

$$\mathbf{c}^{(n+1)} = \varepsilon \mathbf{h}(\mathbf{c}^{(n)}, \varepsilon) \quad (n = 0, 1, \ldots) \tag{4.7}$$

starting with the initial approximation $\mathbf{c}^{(0)}$ (we may take the vector $\mathbf{c}^{(0)} = \mathbf{0}$, which is the solution for $\varepsilon = 0$). For sufficiently small $\varepsilon > 0$, the iterative process is convergent.

Indeed, we see from the proof of Theorem 3 that, for $\varepsilon > 0$ satisfying the condition of the theorem, the norm of the derivative of the operator Φ in Eq.(2.4) is less than 1. Therefore, for sufficiently large m, the norm of the derivative of the finite-dimensional approximation $\varepsilon\mathbf{h}(\mathbf{c}, \varepsilon)$ of the operator Φ in the space of sequences is less than 1.

Therefore, in the metric of the Hermitian space C^{4m+2}, $\varepsilon\mathbf{h}(\mathbf{c}, \varepsilon)$ is a contracting mapping, which ensures convergence of the iterative process to the exact solution of Eq.(4.6).

Thus, assume that we have obtained an approximation $\mathbf{c}^{(n)}$ to the solution of Eq.(4.6) in the space of sequences $\mathbf{C}^{4m+2}$. This means that in the metric of the space $L_2([0, 2\pi], \mathbf{C}^2)$ the function

$$\gamma_{4m+2}^{(n)}(\tau) = \Sigma_m e^{is\tau}\mathbf{c}_s^{(n)} \tag{4.8}$$

approximates the solution (3.3) of Eq.(3.2). But for sufficiently large m the function $\gamma_{4m+2}(\tau)$ in (3.3) approximates by Theorem 3 the exact solution $\gamma(\tau)$ of Eq.(2.4). Therefore, in the interval $[0, 2\pi]$ the function

$$\zeta_{4m+2}^{(n)}(\tau) = \zeta_0 + \tau\mathbf{c}_0^{(n)} + \Sigma_m{}' \frac{e^{is\tau} - 1}{is}\mathbf{c}_s^{(n)} \tag{4.9}$$

ensures uniform approximation of the exact solution of Eq.(1.5). The coefficients $\mathbf{c}_s^{(n)}$ depend on the vector ζ_0 as a parameter. Formula (4.7) enables us to obtain a solution of Eq.(4.6) in both numerical and analytical form.

Formula (4.9) may be applied to construct a Poincaré recurrence mapping. For each initial vector ζ_0, use either a numerical procedure or (4.9) with analytical expressions for the coefficients to construct a path by substituting into (4.9) the coordinates of different initial vectors ζ_0.

5. *Appendix.*

Proof of Theorem 1. The space CA may be decomposed into a direct sum $C^n + CA^\circ$, where CA° consists of functions $\zeta(\tau)$ such that $\zeta(0) = 0$. Then we can show that the differentiation operator restricted to the affine subspace $\zeta_0 + CA^\circ$ is a homeomorphism.

We know that the derivative of an absolutely continuous function is summable, and

$$\mathrm{Var}([a, b], \zeta) = \int_a^b \|(D\zeta)(\tau)\| \, d\tau = \|D\zeta\|_1$$

This property leads to continuity, bijectivity, and openness of the mapping D, i.e., this is a homeomorphism.

Therefore, by fixing the initial vector ζ_0 we fix the affine subspace $\zeta_0 + CA^\circ$, and the solutions of Eq.(1.5) and (2.4) are in one-to-one correspondence by the homeomorphism $D\colon \zeta_0 + CA^\circ \to L_1$.

Proof of Theorem 2. Convergence in L_2 in the finite interval $[a, b]$ implies convergence in L_1. Since D is a homeomorphism on the subspace $\zeta_0 + CA^\circ$ the sequence $\{D^{-1}\gamma_k\}_{k=1}^{\infty}$ converges to the solution $D^{-1}\gamma$ uniformly.

Proof of Theorem 3. In order to apply the results of /3/, we need to check a number of conditions.

First, we need to prove Fréchet-differentiability of the operators Φ and $P_m\Phi$. The mapping Φ is the composition of mappings $\varepsilon\mathbf{Z} \circ D^{-1}$. The affine operator D^{-1} is continuous and therefore differentiable.

Differentiability of $\mathbf{Z}$ can be proved by the Lebesgue theorem on the passage to the limit under the sign of the integral by examining the functional expression

$$\mathbf{Z}(\zeta + \mathbf{h}) - \mathbf{Z}(\zeta) - \mathbf{Z}_\zeta'(\zeta)\cdot\mathbf{h} = \omega(\mathbf{h})$$

The derivative is given by the formula

$$(\mathbf{Z}_\zeta'\mathbf{h})(\tau) = \mathbf{Z}_\zeta'[\zeta(\tau), \tau, \varepsilon]\,\mathbf{h}(\tau) \quad (\mathbf{h} \in CA)$$

The differentiability of $P_m\Phi$ follows from continuity of the projector, and $(P_m\Phi)' = P_m \circ \Phi'$.

Now, we have to show continuous invertibility in the Hilbert space L_2 of the operator $E-\Phi'(\gamma)$, where E is the identity operator. Applying a chain of inequalities, we obtain a bound for the operator norm of $\Phi'(\gamma)$:

$$\|\Phi'(\gamma)\|_{2,2}\leqslant(b-a)^{1/2}\varepsilon\|\mathbf{Z}\|_2^1$$

We see that for sufficiently small ε the operator $E-\Phi'(\gamma)$ is invertible.

Finally, we have to check the approximation conditions. The first condition $\|\gamma-P_m\gamma\|_2\to 0$ $(m\to\infty)$ follows from completeness of the basis functions. The second condition $\|P_m\Phi(P_m\gamma)-\Phi(\gamma)\|_2\to 0$ $(m\to\infty)$ follows from the continuity of the operator Φ. The approximation conditions for the derivatives of the operators $\|P_m\Phi'(P_m\gamma)-\Phi'(\gamma)\|_{2,2}\to 0$ $(m\to\infty)$ are checked as follows. The triangle inequality (and also the bound $\|P_m\|_{2,2}\leqslant 1$) give

$$\|P_m\Phi'(P_m\gamma)-\Phi'(\gamma)\|_{2,2}\leqslant\|\Phi'(P_m\gamma)-\Phi'(\gamma)\|_{2,2}+\|P_m\Phi'(\gamma)-\Phi'(\gamma)\|_{2,2}$$

As before, we use the Lebesgue lemma to prove continuity of the mapping $\Phi'\colon\Omega\to L(L_2)$ to the algebra of linear continuous operators of the space L_2. Therefore $\|\Phi'(P_m\gamma)-\Phi'(\gamma)\|_{2,2}\to 0$ $(m\to\infty)$. Continuity of Φ' is also needed to prove the existence of a solution of Eq.(3.2) by the method of contracting mappings.

Now it is easy to see that the operator $\Phi'(\gamma)$ is compact. Therefore, the set $\Phi'(\gamma)S$, where S is the unit sphere in L_2, is precompact. From pointwise convergence of the sequence of operators $\{P_m-E\}_{m=1}^{\infty}$ we obtain by the Banach-Steinhaus theorem /2/ that the convergence on the set $\Phi'(\gamma)S$ is uniform. Therefore, we finally get

$$\|P_m\Phi'(\gamma)-\Phi'(\gamma)\|_{2,2}=\sup_{\|\mathbf{h}\|_2=1}\|(P_m-E)\Phi'(\gamma)\mathbf{h}\|_2\to 0\ (m\to\infty)$$

All the conditions of /3/ have been checked for Eq.(2.4).

I would like to thank V.G. Demin for his interest.

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Translated by Z.L.